

QUADRATIC FORMS AND THEIR THETA SERIES – INFINITESIMAL ASPECTS

JUAN MARCOS CERVIÑO AND GEORG HEIN

ABSTRACT. We study the theta map which assigns to a real quadratic form its theta series. We introduce two invariants reflecting whether the differential of the theta map vanishes or is degenerate. We provide examples of lattices where this differential is zero. These invariants turn out to be modular forms for integral lattices. We illustrate this in the rank two case.

1. PRELIMINARIES AND NOTATION

Let $\text{Quad}_n(\mathbb{R})$ be the real vector space of quadratic forms in n variables. It is a $\frac{n^2+n}{2}$ -dimensional vector space which we identify with the space of symmetric $n \times n$ matrices with real entries. Inside this vector space is the open cone $\text{Quad}_n^+(\mathbb{R})$ of positive definite forms.

The second space of interest is the ring Mod of holomorphic functions on the upper half plane \mathbb{H} which satisfy a certain growth condition. That is

$$\text{Mod} = \left\{ f : \mathbb{H} \rightarrow \mathbb{C} \left| \begin{array}{l} \text{there exists a discrete subset } M \subset \mathbb{R}_{\geq 0} \\ \text{and a function } a : M \rightarrow \mathbb{C} \text{ with } m \mapsto a_m \\ \text{bounded by two polynomials } H_1 \text{ and } H_2. \\ \text{This means, that for all } d \in \mathbb{R} \text{ we have} \\ \#\{m \in M \mid m \leq d\} \leq H_1(d), \text{ and} \\ \|a_m\| \leq H_2(m) \text{ for all } m \in M. \\ \text{The holomorphic function } f \text{ has an expansion} \\ f(z) = \sum_{m \in M} a_m \exp(2\pi i m z). \end{array} \right. \right\}.$$

Modular forms are typical elements of Mod . Moreover, Mod is closed under differentiation. We are interested in the map Θ which assigns to a quadratic form $A \in \text{Quad}_n^+(\mathbb{R})$ its theta series. With the above notation Θ is a map

$$\Theta : \text{Quad}_n^+(\mathbb{R}) \rightarrow \text{Mod} \quad A \mapsto \Theta_A \quad \text{with } \Theta_A(z) = \sum_{\lambda \in \mathbb{Z}^n} \exp(2\pi i ({}^t \lambda A \lambda) z).$$

Two quadratic forms A and A' are equivalent, if and only if there exists a $T \in \text{GL}_n(\mathbb{Z})$ such that $A' = {}^t T A T$. We call A and A' isospectral when their theta series coincide. Obviously, we have $\Theta_A = \Theta_{A'}$ for two equivalent quadratic forms, in words: equivalence implies isospectrality. The global Torelli theorem for the Θ -map asks, whether we can conclude from $\Theta_A = \Theta_{A'}$ the equivalence of the quadratic forms. A local Torelli statement investigates whether Θ reflects all infinitesimal deformations of a given quadratic form. A. Schiemann showed in [7] that we have a

Global Torelli theorem for the Θ map in small dimensions. *If the rank of two quadratic forms is at most three, then they are isospectral if and only if they are equivalent.*

Moreover, Schiemann gave an example for the failure of a global Torelli for quadratic forms of rank four by providing two inequivalent quadratic forms with the same theta series. Schiemann's example from [6] was generalized to a family of pairs of isospectral quadratic forms by Conway and Sloane in [5]. Conway and Sloane conjectured that pairs of lattices in this family were pairs

of inequivalent lattices. The authors showed in [4] that the invariant $\Theta_{1,1}$ developed in [2] allows to distinguish the two lattices for all non-isometric pairs in the family. Therefore we believe, that if we add to the Θ map some of the invariants developed in [2] and [3], then a global Torelli theorem holds true. We have for example:

Conjecture. *Two quadratic forms A and A' of rank four are equivalent, if and only if $\Theta_A = \Theta_{A'}$ and $\Theta_{1,1;A} = \Theta_{1,1;A'}$ holds.*

In the following Quad_n denotes $\text{Quad}_n(\mathbb{R})$ (equivalently for Quad_n^+). The aim of this paper is to study the infinitesimal behavior of the map Θ , in other words the differential $D\Theta$. For a lattice $A \in \text{Quad}_n^+$ we restrict to the hyperplane T_A^0 of deformations in tangent space which leave the discriminant of A unchanged. In this paper we answer the following questions:

- (1) Are there quadratic forms A such that the differential of the Θ map is zero on the hyperplane T_A^0 in the tangent space?
- (2) For which quadratic forms A is the differential of the Θ map at A degenerate?
- (3) Is the differential of the Θ map injective for a general quadratic form A ?

It turns out that $D\Theta|_{T_A^0}$ is zero, if and only if our invariant $\Theta_{1,1;A}$ vanishes (see Proposition 2.4(iii)). Furthermore, we give examples of quadratic forms with $\Theta_{1,1;A} = 0$ coming from p -th roots of unity (Proposition 3.3), and from large automorphism groups (Proposition 3.5) as root lattices for example. This implies that there cannot be a local Torelli theorem. Thus, the question (3) has a negative answer when we do not restrict to a general quadratic form A . We introduce a new lattice invariant $\det^2 D\Theta$ which is also a modular form for integral quadratic forms and vanishes exactly when the differential $D\Theta$ is degenerate (cf. Proposition 4.1). A simple argument gives the local Torelli theorem (Proposition 4.2) for a general lattice. In the last section we compute these invariants for binary quadratic forms, and identify all the forms with vanishing or degenerate differential.

2. THE METRIC STRUCTURE ON THE TANGENT SPACE OF Quad_n^+

2.1. Quadratic forms and lattices. For a point $A \in \text{Quad}_n^+$ we have a positive definite pairing on the tangent space $T_A = T_{\text{Quad}_n^+, A} = \text{Quad}_n$ given by the Killing form

$$\langle H_1, H_2 \rangle = 2 \cdot \text{tr}(A^{-1}H_1 A^{-1}H_2).$$

Considering \mathbb{R}^n with the scalar product given by A there exists a linear transformation $Q : \mathbb{R}^n \rightarrow \mathbb{R}^n$ from (\mathbb{R}^n, A) to the standard Euclidean space which is an isometry. On the level of matrices we have $A = {}^tQ \cdot Q$. Sometimes we consider instead of A the lattice $\Lambda = \Lambda_A$ generated by the column vectors of Q . The matrix Q is given only up to an element of the orthogonal group $O(n)$. However, when we require Q to be upper triangular with positive diagonal entries, then Q is uniquely given (Cholesky decomposition).

For two quadratic forms h_1 and h_2 on \mathbb{R}^n we have the positive definite pairing $\langle h_1, h_2 \rangle = h_1(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n})h_2(x_1, x_2, \dots, x_n)$. If we represent h_1 and h_2 by symmetric $n \times n$ matrices, then we see that the pairing is given by $\langle h_1, h_2 \rangle = 2 \cdot \text{tr}(h_1 \cdot h_2)$. Since the quadratic form h_i on \mathbb{R}^n corresponds to the quadratic form ${}^tQ^{-1} \cdot h_i \cdot Q^{-1}$ and the trace is invariant under conjugation, we obtain the above positive definite form. Under this identification the quadratic form A on \mathbb{R}^n corresponds to the square length on \mathbb{R}^n .

Thus, H defines a harmonic quadratic form on \mathbb{R}^n with metric given by A if and only if $\text{tr}(A^{-1}H) = 0$.

2.2. The invariant $\Theta_{1,1;A}$. In [2] the authors defined the lattice invariant $\Theta_{1,1;\Lambda}$ for a lattice $\Lambda \subset \mathbb{R}^n$. Since it is invariant under the orthogonal group $O(n)$ it gives an invariant of the

associated quadratic form. For the convenience of the reader we recall its definition (see Theorem 4.2 in [2]).

Let $A : \mathbb{Z}^n \rightarrow \mathbb{R}$ be a positive definite quadratic form, corresponding to the lattice $\Lambda \subset \mathbb{E}^n$. The holomorphic function $\Theta_{1,1;\Lambda} = \Theta_{1,1;\Lambda}$ on the upper half plane is given by

$$\Theta_{1,1;\Lambda}(\tau) = \sum_{m \geq 0} a_m \exp(2\pi i m \tau) \quad \text{with} \quad a_m = \sum_{\substack{(\gamma, \delta) \in \Lambda \times \Lambda \\ \|\gamma\|^2 + \|\delta\|^2 = m}} \left(\frac{\cos^2(\angle(\gamma, \delta))}{2} - \frac{1}{2n} \right) \|\gamma\|^2 \|\delta\|^2.$$

The function $\Theta_{1,1;\Lambda}$ can also be computed using theta series with harmonic coefficients. For a lattice $\Lambda \subset \mathbb{E}^n$ and a harmonic function $h : \mathbb{E}^n \rightarrow \mathbb{C}$ we define

$$\Theta_{h;\Lambda}(\tau) := \sum_{\lambda \in \Lambda} h(\lambda) \exp(2\pi i \|\lambda\|^2 \tau).$$

We can compute $\Theta_{1,1;\Lambda}$ in terms of the functions $\Theta_{h;\Lambda}$ as follows: Let $\{h_i\}_{i=1,\dots,N}$ be an orthonormal basis of the harmonic quadratic forms on \mathbb{E}^n . Then we have an equality (see [3, Theorem 3.3])

$$\Theta_{1,1;\Lambda} = \sum_{i=1}^N \Theta_{h_i;\Lambda}^2.$$

2.3. The splitting of the tangent space T_A . We split the tangent space as follows:

$$T_A = T_{\text{Quad}_n^+, A} = \text{Quad}_n = \mathbb{R} \cdot A \oplus T_A^0 \quad \text{where} \quad T_A^0 = \{A' \in \text{Quad}_n \mid \text{tr}(A' \cdot A^{-1}) = 0\}.$$

The hyperplane T_A^0 describes the infinitesimal deformations with fixed discriminant. We compute

$$\partial_B \Theta_A := \frac{1}{2\pi i z} \frac{\partial}{\partial t} \Theta_{A+tB} \Big|_{t=0} = \sum_{\lambda \in \mathbb{Z}^n} {}^t \lambda B \lambda \exp(2\pi i ({}^t \lambda A \lambda) z).$$

Let A be an integral quadratic form, which means $\Theta_A \in \mathbb{Z}[[q]]$. It follows that Θ_A is a modular form for a subgroup $\Gamma \subset \text{SL}_2(\mathbb{Z})$. For a harmonic quadratic form B we have that $\partial_B \Theta_A$ is a modular form of weight $\frac{n+4}{2}$, whereas $\partial_A \Theta_A = \Theta'_A$. We conclude the following

Proposition 2.4. *Let A be an integer valued quadratic form of rank n and level l .*

- (1) Θ_A is a modular form of weight $\frac{n}{2}$ and level l .
- (2) $\partial_B \Theta_A$ is a modular form of weight $\frac{n+4}{2} \iff B \in T_A^0$. In particular, we have $\partial_B \Theta_A = 0$ only for $B \in T_A^0$.
- (3) The differential $D\Theta$ vanishes identically on $T_A^0 \iff \Theta_{1,1;A} = 0$.

Proof. Only (iii) is new. Let B_1, B_2, \dots, B_N be a orthonormal basis of T_A^0 . The functions $\partial_{B_i} \Theta_A$ take real values on the imaginary axis in the upper half plane. Thus, since harmonic quadratic forms are precisely the elements of T_A^0 (cf. Section 2.1), we have an equivalence

$$\begin{aligned} D\Theta|_{T_A^0} \equiv 0 &\iff \partial_{B_i} \Theta_A = 0 \text{ for all } i = 1, \dots, N \\ &\iff (\partial_{B_i} \Theta_A)^2 = 0 \text{ for all } i = 1, \dots, N \\ &\iff \sum_{i=1}^N (\partial_{B_i} \Theta_A)^2 = 0. \end{aligned}$$

Since $\Theta_{1,1;A} = \sum_{i=1}^N (\partial_{B_i} \Theta_A)^2$, this shows the assertion. \square

3. LATTICES WITH VANISHING DIFFERENTIAL

3.1. First examples for lattices with vanishing differential. Let us start with three quadratic forms with vanishing differential. The vanishing of the modular form $\Theta_{1,1;\Lambda}$ follows in all three cases from Proposition 3.5. Moreover, for the second example it also follows from Proposition 3.3 when setting $p = 3$.

Example 1: The Gaussian integers (A_1^2). The Gaussian integers $\mathbb{Z}[i] \subset \mathbb{C} = \mathbb{E}^2$ form a lattice. The lattice corresponds to the quadratic form $A_1^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Its theta series is given by $\Theta_{A_1^2}(z) = 1 + 4q^1 + 4q^2 + 4q^4 + 8q^5 + 4q^8 + 4q^9 + 8q^{10} + 8q^{13} + 4q^{16} + 8q^{17} + \dots$ with $q = \exp(2\pi iz)$. We have that $\Theta_{1,1;A_1^2} = 0$.

Example 2: The Eisenstein integers (A_2). The Eisenstein integers $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ again form a lattice in \mathbb{E}^2 corresponding to the quadratic form $A_2 = \frac{1}{2} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. We compute its theta series $\Theta_{A_2}(z) = 1 + 6q^1 + 6q^3 + 6q^4 + 12q^7 + 6q^9 + 6q^{12} + 12q^{13} + 6q^{16} + 12q^{19} + \dots$ and $\Theta_{1,1;A_2} = 0$.

Example 3: The E_8 lattice. Here we have $\Theta_{E_8}(z) = 1 + 240q^2 + 2160q^4 + 6720q^6 + 17520q^8 + 30240q^{10} + \dots$. For the unimodular lattice E_8 it was shown in [3, Example 3.4] that $\Theta_{1,1;E_8} = 0$.

3.2. Lattices with vanishing differential from p -th roots of unity. Let p be an odd prime, and $\zeta = \exp(2\pi i/p) \in \mathbb{C}$ be a p -th root of unity, and $K = \mathbb{Q}(\zeta)$. The ring $\mathcal{O}_K = \mathbb{Z}[\zeta]$ possesses $p-1$ embeddings into \mathbb{C} . They form $\frac{p-1}{2}$ conjugated pairs. Choosing one representative from each pair we obtain the Minkowski embedding

$$\iota : \mathcal{O}_K \rightarrow \mathbb{C}^{\frac{p-1}{2}} \quad \text{given by } \zeta^k \mapsto \begin{pmatrix} \zeta^k \\ \zeta^{2k} \\ \vdots \\ \zeta^{\frac{p-1}{2}k} \end{pmatrix}.$$

The embedding of $\mathcal{O}_K \rightarrow \mathbb{C}$ which sends ζ to ζ^k is denoted by σ_k . Let us identify $\mathbb{C}^{\frac{p-1}{2}}$ with the euclidean space \mathbb{E}^{p-1} , and denote by Λ_p the image of \mathcal{O}_K in \mathbb{E}^{p-1} . The lattice Λ_p corresponds to the quadratic form given by

$$(1) \quad A_p = \frac{1}{2} \begin{pmatrix} p-1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & p-1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & p-1 & \cdots & -1 & -1 \\ \vdots & \vdots & & \ddots & & \vdots \\ -1 & -1 & -1 & \cdots & p-1 & -1 \\ -1 & -1 & -1 & \cdots & -1 & p-1 \end{pmatrix}.$$

On $\mathbb{C}^{\frac{p-1}{2}}$ operates the cyclic group G with p elements and generator g acting as multiplication by ζ^l on the l -th components of $\mathbb{C}^{\frac{p-1}{2}}$. Since all eigenvalues of g are different from one we obtain a free action on $\Lambda_p \setminus 0$. Having in mind that Λ_p is the image of \mathcal{O}_K under the Minkowski embedding we obtain an action of the Galois group $G_{K/\mathbb{Q}}$ on Λ_p . Since $G_{K/\mathbb{Q}}$ is abelian we have $\sigma(\iota(x)) = \iota(\sigma(x))$. Using these actions we will show that $\Theta_{1,1;\Lambda_p} = 0$. In order to prepare it,

we compute for $x, y \in \mathcal{O}_K$ the following sum:

$$(2) \quad s(x, y) := \sum_{k=0}^{p-1} \sum_{\sigma \in G_{K/\mathbb{Q}}} \left\langle g^k(\iota(x)), \sigma(\iota(y)) \right\rangle^2.$$

The scalar product for $z = (z_i)_{i=1, \dots, \frac{p-1}{2}}$ and $z' = (z'_i)_{i=1, \dots, \frac{p-1}{2}}$ in $\mathbb{C}^{\frac{p-1}{2}}$ induced by its identification with \mathbb{E}^{p-1} is given by $\langle z, z' \rangle = \frac{1}{2} \sum_{i=1}^{\frac{p-1}{2}} (z_i \overline{z'_i} + \overline{z_i} z'_i)$. We obtain

$$\begin{aligned} s(x, y) &= \sum_{k=0}^{p-1} \sum_{\sigma \in G_{K/\mathbb{Q}}} \left(\frac{1}{2} \sum_{i=1}^{\frac{p-1}{2}} \left(\zeta^{ki} \sigma_i(x) \overline{\sigma_i(\sigma(y))} + \overline{\zeta^{ki} \sigma_i(x)} \sigma_i(\sigma(y)) \right) \right)^2 \\ &= \frac{1}{4} \sum_{k=0}^{p-1} \sum_{\sigma \in G_{K/\mathbb{Q}}} \sum_{i,j=1}^{\frac{p-1}{2}} \zeta^{k(i+j)} \sigma_i(x) \sigma_j(x) \overline{\sigma_i(\sigma(y)) \sigma_j(\sigma(y))} + \\ &\quad \frac{1}{4} \sum_{k=0}^{p-1} \sum_{\sigma \in G_{K/\mathbb{Q}}} \sum_{i,j=1}^{\frac{p-1}{2}} \zeta^{k(i-j)} \sigma_i(x) \overline{\sigma_j(x) \sigma_i(\sigma(y))} \sigma_j(\sigma(y)) + \\ &\quad \frac{1}{4} \sum_{k=0}^{p-1} \sum_{\sigma \in G_{K/\mathbb{Q}}} \sum_{i,j=1}^{\frac{p-1}{2}} \zeta^{k(-i+j)} \overline{\sigma_i(x) \sigma_j(x)} \sigma_i(\sigma(y)) \overline{\sigma_j(\sigma(y))} + \\ &\quad \frac{1}{4} \sum_{k=0}^{p-1} \sum_{\sigma \in G_{K/\mathbb{Q}}} \sum_{i,j=1}^{\frac{p-1}{2}} \zeta^{k(-i-j)} \overline{\sigma_i(x) \sigma_j(x)} \sigma_i(\sigma(y)) \sigma_j(\sigma(y)). \end{aligned}$$

Since for any integer m we have $\sum_{k=0}^{p-1} \zeta^{km} = \begin{cases} p & \text{for } p \mid m \\ 0 & \text{otherwise} \end{cases}$, we conclude that the first and last summands are zero. For the same reason we see that in the second and third summand only for $j = i$ we have a non trivial contribution. This yields

$$s(x, y) = \frac{p}{2} \sum_{\sigma \in G_{K/\mathbb{Q}}} \sum_{i=1}^{\frac{p-1}{2}} \|\sigma_i(x)\|^2 \|\sigma_i(\sigma(y))\|^2.$$

When σ runs through $G_{K/\mathbb{Q}}$ the values $\|\sigma_i(\sigma(y))\|$ run through the set of values $\{\|\sigma_j(y)\|\}_{j=1, \dots, \frac{p-1}{2}}$. Eventually we obtain:

$$(3) \quad s(x, y) = p \sum_{i=1}^{\frac{p-1}{2}} \sum_{j=1}^{\frac{p-1}{2}} \|\sigma_i(x)\|^2 \|\sigma_j(y)\|^2 = p \|\iota(x)\|^2 \|\iota(y)\|^2.$$

Proposition 3.3. *For the lattice Λ_p corresponding to the quadratic form A_p given in (1) the modular form $\Theta_{1,1;\Lambda_p}$ equals zero.*

Proof. We write

$$\Theta_{1,1;\Lambda_p}(z) = \sum_{x,y \in \mathcal{O}_K} \left(\langle \iota(x), \iota(y) \rangle^2 - \frac{1}{p-1} \|\iota(x)\|^2 \|\iota(y)\|^2 \right) q^{\|\iota(x)\|^2 + \|\iota(y)\|^2}.$$

If we consider for a fixed pair $(x, y) \in \mathcal{O}_K$ the sum

$$s := \sum_{g \in G} \sum_{\sigma \in G_{K/\mathbb{Q}}} \langle g(\iota(x)), \iota(\sigma(y)) \rangle^2 - \frac{1}{p-1} \|g(\iota(x))\|^2 \|\iota(\sigma(y))\|^2,$$

then, since both groups act as isometries, we obtain using the definition of $s(x, y)$ in (2):

$$s = s(x, y) - p\|\iota(x)\|^2\|\iota(y)\|^2.$$

Our formula (3) for $s(x, y)$ shows that $s = 0$. □

3.4. Large isometry groups imply the vanishing of $\Theta_{1,1;\Lambda}$. The vanishing of $\Theta_{1,1;\Lambda_p}$ in Proposition 3.3 is due to the fact, that the lattice Λ_p has “enough” automorphisms. More precisely, we have the following general result.

Proposition 3.5. *Let $\Lambda \subset \mathbb{E}^n$ be a lattice and suppose its orthogonal group $O_\Lambda := \{\varphi \in O(n) \mid \varphi(\Lambda) = \Lambda\}$ acts irreducibly on \mathbb{E}^n . Then $\Theta_{1,1;\Lambda}$ is zero.*

Proof. Let $G = O_\Lambda$ be the orthogonal group of Λ . We consider the following function

$$Q : \mathbb{E}^n \times \mathbb{E}^n \rightarrow \mathbb{R} \quad (x, y) \mapsto \sum_{g \in G} \langle g(x), y \rangle^2.$$

Since \mathbb{E}^n is an irreducible representation of G , a vector $y \neq 0$ cannot be orthogonal to all vectors $\{g(x)\}_{g \in G}$ unless $x = 0$. We conclude, that for fixed $y \neq 0$ in \mathbb{E}^n the quadratic form $Q_y : \mathbb{E}^n \rightarrow \mathbb{R}$ which sends $x \mapsto Q(x, y)$ is positive definite. By definition Q_y is G -invariant, i.e. $Q_y(g(x)) = Q_y(x)$. The irreducibility of the G -action on \mathbb{E}^n implies that Q_y is proportional to the euclidean quadratic form $x \mapsto \|x\|^2$. Thus, $Q_y(x) = \varphi(y) \cdot \|x\|^2$ for some real number $\varphi(y)$ depending on y .

Analogously, we can define for a vector $x \in \mathbb{E}^n$ the quadratic form $Q_x(y) = Q(x, y)$. The group G is a subgroup of $O(n)$, hence $Q_x(gy) = Q_x(y)$. Eventually, we deduce that $Q(x, y) = c \cdot \|x\|^2 \cdot \|y\|^2$ for some constant $c \in \mathbb{R}$. We compute the constant c by averaging x over all vectors in the unit sphere S^{n-1} in \mathbb{E}^n , and taking $y = {}^t(1, 0, \dots, 0)$. On S^{n-1} we consider the unique invariant measure $d\mu$ such that $\int_{S^{n-1}} d\mu = 1$, and so

$$\begin{aligned} c &= \int_{S^{n-1}} \sum_{g \in G} \langle g(x), y \rangle^2 d\mu(x) \\ &= \sum_{g \in G} \int_{S^{n-1}} \langle g(x), y \rangle^2 d\mu(x) && \text{since } G \text{ acts isometrically we have} \\ &= |G| \int_{S^{n-1}} \langle x, y \rangle^2 d\mu(x) \\ &= |G| \int_{S^{n-1}} x_1^2 d\mu(x) && \text{since } \sum_{i=1}^n x_i^2 = 1 \text{ we conclude that} \\ &= \frac{|G|}{n}. \end{aligned}$$

Using this expression we deduce from

$$\Theta_{1,1;\Lambda}(z) = \sum_{\lambda \in \Lambda} \sum_{\mu \in \Lambda} \left(\langle \lambda, \mu \rangle^2 - \frac{1}{n} \|\lambda\|^2 \|\mu\|^2 \right) q^{\|\lambda\|^2 + \|\mu\|^2}$$

the formula

$$\Theta_{1,1;\Lambda}(z) = \sum_{[\lambda] \in \Lambda/G} \sum_{\mu \in \Lambda} \frac{1}{\#\text{Stab}(\lambda)} \sum_{g \in G} \left(\langle g(\lambda), \mu \rangle^2 - \frac{1}{n} \|g(\lambda)\|^2 \|\mu\|^2 \right) q^{\|g(\lambda)\|^2 + \|\mu\|^2}$$

by considering the G -orbits of the element $\lambda \in \Lambda$. The sum over G is already zero – which follows immediately from $g \in G \subset O(n)$ and $c = |G|/n$ –, and therefore $\Theta_{1,1;\Lambda} \equiv 0$. □

Remark 1. In his diploma thesis T. Alfs gives a formula for the invariant $\Theta_{1,1;\Lambda}$ when $\Lambda = \Lambda_1 \oplus \Lambda_2$ is a direct sum of two lattices. He shows in [1, Satz 6.3] that we have

$$\Theta_{1,1;\Lambda_1 \oplus \Lambda_2} = \frac{n_1}{n_1 + n_2} \Theta_{1,1;\Lambda_1} \Theta_{\Lambda_2}^2 + \frac{n_2}{n_1 + n_2} \Theta_{1,1;\Lambda_2} \Theta_{\Lambda_1}^2 + \frac{2}{n_1 n_2 (n_1 + n_2)^2} F_1(\Theta_{\Lambda_1}, \Theta_{\Lambda_2})^2$$

where Λ_i is a lattice in \mathbb{E}^{n_i} and $F_1(\Theta_{\Lambda_1}, \Theta_{\Lambda_2})$ is the first Rankin–Cohen differential operator see [8, Section 1.3].

Remark 2. As a consequence we obtain that for a lattice Λ with $\Theta_{1,1;\Lambda} = 0$ we also have $\Theta_{1,1;\Lambda^{\oplus m}} = 0$ for all $m \geq 1$. Using Proposition 3.5 we deduce the vanishing of $\Theta_{1,1}$ for all powers of root lattices A_n^k, D_n^k, E_n^k .

4. LATTICES WITH DEGENERATE DIFFERENTIAL

Let now $\Lambda \subset \mathbb{E}^n$ be a lattice with corresponding quadratic form A . Let $\mathcal{B} = \{h_i\}_{i=1,\dots,m}$ with $m = \frac{n^2+n-2}{2}$ be an orthonormal basis of the vector space of homogeneous harmonic polynomials of degree two. We set $\Theta_{h_i}(\Lambda) := \sum_{\lambda \in \Lambda} h(\lambda) q^{\|\lambda\|^2}$.

We have seen in Section 2.3 that the vectors $\Theta_{h_i}(\Lambda)$ in the vector space Mod are (up to the scaling factor $2\pi iz$) the differentials of a basis of the tangent space T_A^0 of A in Quad_n^+ . These functions are linearly dependent if and only if the following determinant is zero:

$$\det_{\mathcal{B}} D\Theta_{\Lambda}(z) := \det \begin{pmatrix} \Theta_{h_1} & \Theta_{h_2} & \cdots & \Theta_{h_m} \\ \frac{\partial}{2\pi i \partial z} \Theta_{h_1} & \frac{\partial}{2\pi i \partial z} \Theta_{h_2} & \cdots & \frac{\partial}{2\pi i \partial z} \Theta_{h_m} \\ \left(\frac{\partial}{2\pi i \partial z}\right)^2 \Theta_{h_1} & \left(\frac{\partial}{2\pi i \partial z}\right)^2 \Theta_{h_2} & \cdots & \left(\frac{\partial}{2\pi i \partial z}\right)^2 \Theta_{h_m} \\ \vdots & \vdots & \ddots & \vdots \\ \left(\frac{\partial}{2\pi i \partial z}\right)^{m-1} \Theta_{h_1} & \left(\frac{\partial}{2\pi i \partial z}\right)^{m-1} \Theta_{h_2} & \cdots & \left(\frac{\partial}{2\pi i \partial z}\right)^{m-1} \Theta_{h_m} \end{pmatrix}.$$

The scaling factor $\frac{1}{2\pi i}$ is included to make differentiation closed in the ring $\mathbb{Z}[[q]]$. When Λ is an integral lattice, then it turns out that $\det_{\mathcal{B}} D\Theta_{\Lambda}$ is a modular form. Suppose that Λ is integral of level N . Then the Θ_{h_i} are modular forms of weight $w = \frac{n}{2} + 2$ for the group $\Gamma_0(N)$. We write $\Theta_{\underline{h}}$ for the row vector $(\Theta_{h_1}, \dots, \Theta_{h_m})$. We deduce inductively from

$$\Theta_{\underline{h}}(\gamma(z)) = (cz + d)^w \Theta_{\underline{h}}(z) \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),$$

that

$$\frac{\partial^k \Theta_{\underline{h}}}{\partial z^k}(\gamma(z)) = (cz + d)^{w+2k} \frac{\partial^k \Theta_{\underline{h}}}{\partial z^k}(z) + \sum_{l=0}^{k-1} a_{l,k} (cz + d)^{w+k+l} \frac{\partial^l \Theta_{\underline{h}}}{\partial z^l}(z)$$

with constants $a_{l,k}$ depending only on l, k, w and c . Therefore, modulo the span of

$$\left\{ (cz + d)^{w+k+l} \frac{\partial^l \Theta_{\underline{h}}}{\partial z^l} \mid l = 0, \dots, k-1 \right\}$$

the vector $\frac{\partial^k \Theta_{\underline{h}}}{\partial z^k}$ behaves like a modular form of weight $w + 2k$. When computing the determinant above, we may hence assume that $\frac{\partial^k \Theta_{\underline{h}}}{\partial z^k}$ is a modular form of weight $w + 2k$ since adding of rows does not alter the determinant. We conclude that $\det_{\mathcal{B}} D\Theta_{\Lambda}$ is a modular form of weight

$m(w + m - 1) = \frac{(n+2)^2 n(n-1)}{4}$. Since changing the orthonormal basis \mathcal{B} multiplies the original determinant by ± 1 , we define $\det^2 D\Theta_\Lambda := (\det_{\mathcal{B}} D\Theta_\Lambda)^2$ for any orthonormal basis \mathcal{B} of degree two harmonic polynomials and obtain

Proposition 4.1. *The function $\det^2 D\Theta_\Lambda$ is a lattice invariant. We have an equivalence*

$$\det^2 D\Theta_\Lambda = 0 \iff \text{the differential of the theta map is degenerated at } \Lambda.$$

If Λ is an integral lattice of level N , then $\det^2 D\Theta_\Lambda$ is a modular form of weight $\frac{(n+2)^2 n(n-1)}{2}$ for $\Gamma_0(N)$.

We will write $\det D\Theta_\Lambda$ for $\det_{\mathcal{B}} D\Theta_\Lambda$ whenever the basis \mathcal{B} is understood from the context or is irrelevant. For example when considering whether $\det D\Theta_\Lambda = 0$.

Remark 3. *Similar to the constructions in Section 3 we can now construct lattices Λ such that $\det D\Theta_\Lambda = 0$. Geometrically speaking, these are the lattices where the local Torelli theorem fails for the theta map. Since all the examples with vanishing differential from Section 3 give also lattices with degenerate differential we give here only one example:*

Let $\Lambda \subset \mathbb{E}^n$ be a lattice that is invariant under a reflection $\rho : \mathbb{E}^n \rightarrow \mathbb{E}^n$. Then we have $\det D\Theta_\Lambda = 0$. For a proof of this assertion take two coordinates x_1 and x_2 on \mathbb{E}^n such that $\rho^ x_i = (-1)^i x_i$. Then $\Theta_{x_1 x_2; \Lambda} = 0$, and so is $\det D\Theta_\Lambda = 0$.*

Proposition 4.2. (Generic local Torelli theorem) *For a general lattice $\Lambda \subset \mathbb{E}^n$ we have $\det D\Theta_\Lambda \neq 0$. This means, the Θ -map is generically locally injective.*

Proof. It is enough to show that there exists a lattice Λ such that the differential of the Θ -map is of full rank in Λ . This will imply the statement of the proposition.

We start with a positive quadratic form $A \in \text{Quad}_n^+$ given by a symmetric matrix $A = (a_{ij})_{i,j=1,\dots,n}$. We assume furthermore that the set $\{a_{ij}\}_{1 \leq i \leq j \leq n}$ is linear independent in the \mathbb{Q} vector space \mathbb{R} . This linear independence implies by definition that for two vectors \underline{m} and \underline{m}' in \mathbb{Z}^n we can conclude from $A(\underline{m}) = A(\underline{m}')$ that $\underline{m} = \pm \underline{m}'$.

Let now $\Lambda \subset \mathbb{E}^n$ be the lattice associated to A . We claim that $\det D\Theta_\Lambda \neq 0$. On the contrary, assume that $\det D\Theta_\Lambda = 0$. This implies by Proposition 4.1 that the functions $\{\Theta_{h_i; \Lambda}\}$ are linearly dependent where the h_i form a basis of the harmonic homogeneous quadratic forms on \mathbb{E}^n . Therefore there exists a harmonic quadratic form h such that $\Theta_{h; \Lambda} = 0$. Since for two $\lambda, \lambda' \in \Lambda$ we have $\|\lambda\|^2 = \|\lambda'\|^2$ only for $\lambda' = \pm \lambda$, we conclude that the coefficient of $\exp(2\pi i \|\lambda\|^2 z)$ in $\Theta_{h; \Lambda}$ is $h(\lambda) + h(-\lambda) = 2h(\lambda)$. This implies $h|_\Lambda \equiv 0$, and thus $h = 0$. Hence $D\Theta_\Lambda$ has full rank. \square

5. EXAMPLE: LATTICES IN DIMENSION TWO

5.1. Lattices with vanishing differential in dimension 2.

Proposition 5.2. *The above two examples – the Gaussian and the Eisenstein integers – are up to scalar multiples the only examples of lattices with vanishing Θ -differential in dimension two.*

Proof. We consider the lattice vectors $\lambda \neq 0$ of minimal length in our lattice $\Lambda \subset \mathbb{E}^2$. These appear pairwise, since $-\lambda$ and λ have the same length. Furthermore, we scale Λ such that $\|\pm \lambda\| = 1$ holds for a pair of vectors of minimal positive length. There are three cases:

Case 1: There exists one pair $(\lambda, -\lambda)$ of vectors of minimal length. The coefficient c_2 of q^2 in $\Theta_{1,1; \Lambda}$ is given by

$$c_2 = 4(2 \langle \lambda, \lambda \rangle^2 - \|\lambda\|^4) = 4.$$

Thus, $\Theta_{1,1; \Lambda}$ can not be zero.

Case 2: There exists two pairs $(\lambda, -\lambda)$ $(\mu, -\mu)$ of vectors of minimal length. Again we compute the coefficient c_2 of q^2 to be

$$c_2 = 4(2\langle \lambda, \lambda \rangle^2 - \|\lambda\|^4) + 4(2\langle \mu, \mu \rangle^2 - \|\mu\|^4) + 8(2\langle \lambda, \mu \rangle^2 - \|\lambda\|^2\|\mu\|^2) = 16\langle \lambda, \mu \rangle^2.$$

Thus, we see that for $\Theta_{1,1;\Lambda} = 0$ we must have $\langle \lambda, \mu \rangle = 0$. This is the case of the Gaussian integers (Example 1 in §3.1).

Case 3: There exists three pairs of vectors of minimal length. Indeed this happens only for the Eisenstein lattice (Example 2 in §3.1). \square

5.3. Lattices with degenerate differential in dimension two. We will now investigate for which lattices in dimension two the differential is degenerate. We consider the harmonic functions $h_1(x, y) = x^2 - y^2$ and $h_2(x, y) = xy$. They form a orthogonal basis of the harmonic functions of degree two in two variables. Thus, for a lattice $\Lambda \subset \mathbb{E}^2$ the differential of the Θ -map (restricted to T_A^0) is spanned by Θ_{h_1} and Θ_{h_2} with

$$\Theta_{h_i} = \sum_{\lambda \in \Lambda} h_i(\lambda) q^{\|\lambda\|^2}.$$

The differential degenerates whenever the difference $\Theta_{h_1}\Theta'_{h_2} - \Theta'_{h_1}\Theta_{h_2}$ is zero. We compute hence this function, which is the function $\det D\Theta_\Lambda$ with the appropriate scaling (cf. Section 4):

$$\det D\Theta_\Lambda = \frac{1}{2\pi i}(\Theta_{h_1}\Theta'_{h_2} - \Theta'_{h_1}\Theta_{h_2}) = \sum_{(\lambda, \mu) \in \Lambda^2} h_1(\lambda)h_2(\mu)(\|\mu\|^2 - \|\lambda\|^2)q^{\|\lambda\|^2 + \|\mu\|^2}.$$

We can symmetrize this expression by interchanging the lattice elements λ and μ in this summation to obtain:

$$2\det D\Theta_\Lambda = \sum_{(\lambda, \mu) \in \Lambda^2} (h_1(\lambda)h_2(\mu) - h_1(\mu)h_2(\lambda))(\|\mu\|^2 - \|\lambda\|^2)q^{\|\lambda\|^2 + \|\mu\|^2}.$$

Expanding these terms yields

$$(4) \quad \det D\Theta_\Lambda = \frac{1}{2} \sum_{(\lambda, \mu) \in \Lambda^2} \langle \lambda, \mu \rangle \det(\lambda, \mu)(\|\mu\|^2 - \|\lambda\|^2)q^{\|\lambda\|^2 + \|\mu\|^2},$$

where for $\lambda = (\lambda_1, \lambda_2)$, and $\mu = (\mu_1, \mu_2)$ we write $\det(\lambda, \mu) := \lambda_1\mu_2 - \lambda_2\mu_1$ for the usual determinant. Whereas $\langle \lambda, \mu \rangle$, $\|\lambda\|^2$, and $\|\mu\|^2$ are $O(2)$ -invariants the determinant transforms like $\det(\gamma(\lambda), \gamma(\mu)) = \det(\gamma) \det(\lambda, \mu)$. Thus $\det D\Theta_\Lambda$ is a half-invariant for the orthogonal group, since $\det D\Theta_{\gamma(\Lambda)} = \det(\gamma) \det D\Theta_\Lambda$. By Proposition 4.1 $\det D\Theta_\Lambda$ is zero if and only if the differential of the Θ map is degenerate. Next we determine its degeneration locus.

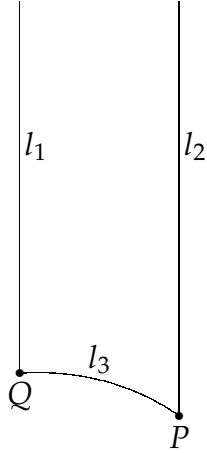
Lemma 5.4. *For a lattice $\Lambda \subset \mathbb{E}^2$ we have $\det D\Theta_\Lambda = 0$ in the following three cases:*

- (1) Λ is spanned by two orthogonal vectors.
- (2) Λ is spanned by a pair (λ_1, λ_2) of vectors with $2\langle \lambda_1, \lambda_2 \rangle = \|\lambda_1\|^2$.
- (3) Λ is spanned by two vectors of the same length.

Proof. In all three cases we use the following fact: If there exists an element $\gamma \in O(2)$ of determinant $\det(\gamma) = -1$ such that $\gamma(\Lambda) = \Lambda$, then the formula $\det D\Theta_{\gamma(\Lambda)} = \det(\gamma) \det D\Theta_\Lambda$ implies $\det D\Theta_\Lambda = 0$. Identifying \mathbb{E}^2 with the complex numbers \mathbb{C} the maps γ are reflections r_τ with $r_\tau(z) = \frac{\tau\bar{z}}{\bar{\tau}}$ is the reflection on the real line generated by τ .

- (1) If $\Lambda = \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2$ with $\lambda_1 \perp \lambda_2$, then we can take the reflection r_{λ_1} or r_{λ_2} .
- (2) If $\Lambda = \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2$ with $2\langle \lambda_1, \lambda_2 \rangle = \|\lambda_1\|^2$, then the reflection r_{λ_1} preserves Λ .
- (3) If $\Lambda = \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2$ with $\|\lambda_1\| = \|\lambda_2\|$, then the reflection $r_{\lambda_1 + \lambda_2}$ interchanges λ_1 and λ_2 .

Thus in all the above cases we see that $\det D\Theta_\Lambda = 0$. If Λ is a lattice which does not belong to the three cases above, then we see that for the pairs $0, (\pm\lambda_1), (\pm\lambda_2), (\pm\lambda_3), \dots$ of lattice vectors ordered by length we have strict inequalities $0 < \|\lambda_1\| < \|\lambda_2\| < \|\lambda_3\|$. Using the q -expansion of $\det D\Theta_\Lambda$ given in (4) we can compute the coefficient of $q^{\|\lambda_1\|^2 + \|\lambda_2\|^2}$ to be $2 \langle \lambda_1, \lambda_2 \rangle \det(\lambda_1, \lambda_2) (\|\lambda_2\|^2 - \|\lambda_1\|^2)$. In this remaining case, each of these three factors (besides 2) are non zero, giving $\det D\Theta_\Lambda \neq 0$. \square



$$F = \left\{ \tau \in \mathbb{H} \text{ with } \|\tau\| \geq 1, \text{ and } 0 \leq \operatorname{Re}(\tau) \leq \frac{1}{2} \right\}$$

The differential of the Θ map is degenerate at $\tau \in F \iff \tau \in (l_1 \cup l_2 \cup l_3)$.

The lattices parameterized by l_i correspond to the case (i) in Lemma 5.4. For example, the lattices parameterized by $\tau \in l_1$ are spanned by two orthogonal vectors.

The differential of the Θ map maximally degenerates at the two points P and Q .

Figure: The fundamental domain F for lattices in the upper half plane \mathbb{H} .

5.5. The weak local Torelli theorem. The vanishing of $\det D\Theta_\Lambda$ along the curves l_i may at a first glance be interpreted as a failure of the local Torelli theorem. However, any continuous lattice invariant on the upper half plane \mathbb{H} with values in a vector space V must send the two lattices generated by $(1, a\sqrt{-1} \pm b)$ for real a and b to the same element of V , independent of the sign. Thus, the differential of the invariant must be degenerate at the point $(1, a\sqrt{-1})$. Hence the differential is degenerate along l_1 . Analogously we see, that the differential must be degenerate along l_2 and l_3 .

The global Torelli theorem is an elementary exercise for rank two lattices. Indeed, the first three lengths of lattice vectors and their multiplicities give the Gram matrix of the lattice.

Acknowledgment. This work has been supported by the SFB/TR 45 “Periods, moduli spaces and arithmetic of algebraic varieties”.

REFERENCES

- [1] T. Alfs, *Die direkte Summenformel für $\Theta_{1,1}$* , Diploma thesis, 2011, Universität Duisburg-Essen.
- [2] J.M. Cerviño, and G. Hein, *Lattice invariants from the heat kernel*, preprint 2009, math.NT/0906.1128.
- [3] J.M. Cerviño, and G. Hein, *Lattice invariants from the heat kernel II*, preprint 2009, math.NT/0909.0340.
- [4] J.M. Cerviño, and G. Hein, *The Conway-Sloane tetralattice pairs are non-isometric*, Advances in Mathematics (2011), doi:10.1016/j.aim.2011.05.016.
- [5] J.H. Conway, and N.J.A. Sloane, *Four-dimensional lattices with the same theta series*, Int. Math. Res. Notices **4** (1992) 93–96.
- [6] A. Schiemann, *Ein Beispiel positiv definiter quadratischer Formen der Dimension 4 mit gleichen Darstellungszahlen* Arch. Math. **54** (1990) 372–375.
- [7] A. Schiemann, *Ternary positive definite quadratic forms are determined by their theta series*, Math. Ann. **308** (1997) 507–517.
- [8] D. Zagier, *Introduction to modular forms* in: M. Waldschmidt, P. Moussa, J.M. Luck and C. Itzykson (eds.) From number theory to physics (Les Houches, 1989), 238–291, Springer, Berlin, 1992.

MATHEMATISCHES INSTITUT, UNIVERSITÄT HEIDELBERG, 69120 HEIDELBERG, GERMANY
E-mail address: `juan.cervino@iwr.uni-heidelberg.de`

FAKULTÄT MATHEMATIK, UNIVERSITÄT DUISBURG-ESSEN, 45117 ESSEN, GERMANY
E-mail address: `georg.hein@uni-due.de`